On the problem of the densest packing of spherical segments into a sphere

Sobre o problema do empacotamento mais denso de segmentos esféricos em uma esfera

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Abstract

The paper considers a particular variant of the classical optimal packing problem when the container is a sphere, the packed elements are equal spherical caps, and the optimality criterion is to maximize their geodesic radius. At the same time, we deal with a special integral metric to determine the distance between points, which becomes Euclidean in the simplest case. We propose a heuristic numerical algorithm based on the construction of spherical Voronoi diagrams, which makes it possible to obtain a locally optimal solution to the problem under consideration. Numerical calculations show the operability and effectiveness of the proposed method and allow us to draw some conclusions about the properties of packings.

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Resumo
O artigo considera uma variante particular do clássico problema de empacotamento ótimo quando o recipiente é uma esfera, os elementos empacotados são calotas esféricas iguais e o critério de otimalidade é maximizar seu raio geodésico. Ao mesmo tempo, tratamos de uma métrica integral especial para determinar a distância entre pontos, que se torna euclidiana no caso mais simples. Propomos um algoritmo numérico heurístico baseado na construção de diagramas esféricos de Voronoi, que permite obter uma solução local ótima para o problema em consideração. Cálculos numéricos mostram a operabilidade e eficácia do método proposto e permitem tirar algumas conclusões sobre as propriedades dos recheios.


Introduction

Approximation of a set by unifying elements is one of the important approaches to solving a number of problems in communication, economics, and logistic (Bychkov, Kazakov, Lempert, 2016). One of the approaches to solve the problem of approximation of a flat compact set is optimal packing. The problem of optimal packing by circles has been studied by many authors using different methods such as packing convex figures (squares, convex polygons, circles) and non-convex figures (Takhonov, 2014; Dorninger, 2017). In fact, the solution of the optimal packing problem is widely used in communication. In telecommunication signal processing, the use of spherical codes is considered to be a special method for increasing noise immunity in long-distance data transmission and storage, for example, when transmitting signals from Earth to satellite and vice versa (Bykhovsky, 2018). Spherical codes were introduced in 1948 by author Shannon C.E. in his article "A Mathematical Theory of Communications" (Shannon, 1948). This type of code used the method of packing the surface of a unit sphere with spheres of a given radius r in n-dimensional space. A detailed description of spherical codes is presented in (Chernyshev, 2008). To improve noise immunity and minimize errors that may occur during transmission,
it is necessary to increase the number of packing spherical segments on the unit sphere, respectively the radius of these segments will decrease. This problem in mathematics corresponds to the Tammees problem given by the Dutch botanist R. M. L. Tammees in 1930.

The problem Tammees (Tammees, 1930) posed in his work "On the origin of number and arrangement of the places of exit on the surface of pollen-grains" consists in arranging n given circles on the surface of a sphere such that the distance between the circles is minimal. This is a special form of Thomson's general problem, which has been studied by a number of mathematicians since the 1940s in more specific cases (Dorninger, 2017). So far, this problem has been solved under the conditions of all values of n < 15 and n = 24. L. Fejes Toth solved this problem in 1943 for n = 3, 4, 6, 12 see O. Toth, 1943 & 1953), C. Schütte and B. van der Varden in 1951 for n = 5, 7, 8, 9 (Schutte & Waerden, 1951) L. Danzer in 1963 for n = 10, 11 (Danzer, 1986), Musin O. R. and A. S. Tarasov in 2012 for n = 13, 14 (Musin, A.S.Tarasov, 2012 & 2015), R. M. Robinson in 1961 for n = 24 (Robinson, 1986).

For other values of n, the problem is not solved, the studies only give an estimate of the possible limit of the radius of spherical segments, and the results obtained are relative. In addition to these, the optimal packing problem is widely used in the distribution of logistics centers (Bychkov & Kazakov, 2016), so the problem needs to be solved not only for large values of n, but also in metric space.

Within the framework of this paper, the optical-geometric approach is used to construct an algorithm for solving the problem of packing three-dimensional bodies with spherical segments.

**Formulation**

Let we are given a metric space \( X \), a bounded surface \( S \subset X \), and a function \( 0 \leq f(x, y, z) \leq \beta \) which define an instantaneous speed of moving for every point \((x, y, z) \in S\). If \( f(x_i, y_i, z_i) = 0 \), then the point \((x_i, y_i, z_i)\) is impassible. We consider the distance between two any points as follow:

\[
\rho(a, b) = \min_{\Gamma \in G(a, b)} \int_{\Gamma} \frac{d\Gamma}{f(x, y, z)} \tag{2.1}
\]

Here \( G(a, b) \) is the set of continuous curves that belong to \( S \) and connect the points \( a \) and \( b \). In this case, the best (fastest) path between two points is a curve that requires the
least time to be passed (Kazakov & Lempert, 2011). If \( f(x_i, y_i, z_i) = 1 \), then \( \rho(a, b) \) is also the minimum distance between points \( a \) and \( b \) (shortest path).

It is necessary to find such an arrangement of \( n \) spherical caps \( C_i(O_i) \) having the pole \( O_i = (x_i, y_i, z_i), i = 1, ..., n \), and geodesic radius \( R \) that provides maximum \( R \) and avoids the caps’ intersection.

\[
R \rightarrow \max \quad (2.2)
\]
\[
\rho(O_i, O_j) \geq 2R, \forall i, j = 1, n, i \neq j \quad (2.3)
\]
\[
O_i \in S, i = 1, n \quad (2.4)
\]

The objective function (2.2) maximizes the radius. Condition (2.3) guarantees that the caps do not intersect each other, and condition (2.4) is that all poles belong to packed sphere \( S \). Figure 1 shows the spherical cap \( C_i \), its pole \( O_i \) and geodesic radius \( R \) (bold line).

![Figure 1. Spherical cap \( C_i \).](source: Authors findings)

**Solution Method**

The basic idea is to improve gradually a randomly generated set of spherical cap centers using the concept of Chebyshev center (Kazakov & Lebedev, 2008) and Voronoi diagram.

First, we construct the Voronoi diagram for a randomly generated initial set of cap centers. Second, for each Voronoi cell, we find the Chebyshev center, which is also the center of the packed spherical cap with the maximum radius. The next step is to re-construct the
Voronoi diagram relative to the set of found centers. The process repeats until the distance between the found and a new center becomes less than a specified small parameter.

3.1 Construction of the Voronoi Diagram

The Voronoi diagram, named after the Russian mathematician Georgy Voronoi, is a method of partitioning a finite set of points S, in which each area of this partition forms a set of points that are closer to one of the elements of the set S than to any other element of the set (Fejes Tóth, 1976). For each point \( O_i = (x_i, y_i, z_i) \in S, i = 1, n \), the Voronoi cell \( V_i \) is as follow (Takhonov, 2014):

\[
V_i = \{ a \in S \mid \rho(a, O_i) \leq \rho(a, O_j), \forall j \neq i \}.
\]

The Voronoi diagram on a plane can be constructed using, for example, well-known Fortune’s algorithm (Fortune, 1987) but it does not allow dealing with a spherical surface. To construct an analogue of the Voronoi diagram on a spherical surface, we use the optical-geometric approach (Kazakov & Lempert, 2011 & 2023). It is based on variational principles of mechanics (Lančzo, 1965), namely according to the physical principles of Fermat and Huygens (Feynman, 1977).

To begin with, let us move on to spherical coordinates. Then the sphere \( x^2 + y^2 + z^2 = r^2 \) takes the form

\[
\Theta = \{ (\alpha, \beta) \mid 0 \leq \alpha < 360, -90 \leq \beta \leq 90 \},
\]

Where:

\( \alpha \) is longitude and \( \beta \) is latitude. Then \( x, y, z \) takes the form 0:

\[
\begin{align*}
x &= r \cos \beta \cos \alpha, \\
y &= r \cos \beta \sin \alpha, \\
z &= r \sin \beta.
\end{align*}
\]
It follows from the geometry of the sphere that the arc $ab$ connecting the points $a$ and $b$ is a curve of minimum length $d(a, b) = \arccos(\overrightarrow{a} \cdot \overrightarrow{b})$, where $\overrightarrow{a} \cdot \overrightarrow{b}$ is scalar product of vectors $Oa$ and $Ob$. Besides, we need to define the distance between a point $C$ and a curve $AB$:

$$d(C, AB) = \min_{X \in AB} d(C, X).$$

**Algorithm of Voronoi Spherical Diagram Construction**

Step 1: A uniform grid is introduced with a stride grid $h$: $S_h \in S$

Step 2: For each point $O_i \in S_h$, $i = 1, n$, the light wave is released and the time to reach $T_i(s)$ all points $s(\alpha, \beta) \in S_h$ is determined. This allows us to find the vector $T(s) = \{T_i(s), i = 1, n\}$

Step 3: For each point $s \in S_h$ the number of the wave, possibly not the only one, which first reached this point is determined by the set $D(s) = \{k: T_k(s) = \min_i T_i(s)\}$

Step 4: We are defined the Dirichlet regions of the Voronoi diagram $V_i$ with center $O_i$ as $V_i = \{s \in S_h: i \in D(s)\}$

**4.1 Packing to the Voronoi Cell**

The Voronoi cells are spherical polyhedra. Let us pack a spherical segment with a maximum radius. Unlike the covering problem, where we find the Chebyshev center of the set, in other words, determine the distance between the points; in the packing problem, it is necessary to determine the distance between the point and the edge. Here we propose an algorithm for finding the center and radius of a packed spherical segment using an optical-geometric approach.

The key idea is as follows. For each Voronoi cell, we find an analogue of Chebyshev center (Vandenberghe, 2004) with respect to metric (2.1). This point gives us the center of the maximum packed spherical cap. Next, using all these centers, we re-construct the Voronoi diagram as long as the radius continues increasing (see, Fig. 2).
Packing Construction Algorithm

Step 0: Introduce a uniform grid with step $h$: $S_h \in S$.

Step 1: Randomly generate initial geodesic center coordinates of spherical caps $O_i \in S_h, i = 1, n$.

Step 2: For the set $C = \{O_i \in S_h, i = 1, n\}$, construct Voronoi cells $V_i, i = 1, n$.

Step 3: Determine the boundary $\partial V_i$ of the cell $V_i$ and approximate it by a closed polyline with nodes at points $v_{i,k}, k = 1, m$. Thus, we obtain sets

$$P(\partial V_i) = \{v_{i,k} | k = 1, m\}.$$ 

Step 4: From points $s(\alpha, \beta) \equiv O_i$ initiate light waves and determine the minimal time that require to reach $P(\partial V_i)$:

$$\rho(s, \partial V_i) = \min_{k=1,m} \rho(s, v_{i,k}).$$

Step 5: The set of closest points to the point $s(\alpha, \beta)$ is determined as follows: $\Delta s = \{(\alpha + \Delta, \beta + \Delta) | \Delta \in (-h, 0, h)\}$.

Step 6: Calculate $\rho(s_{\text{new}}, \partial V_i)$ for each point $s_{\text{new}} \in \Delta s$. If $\min_{s_{\text{new}}, s_{\text{old}}} \rho(s_{\text{new}}, \partial V_i) > \rho(s, \partial V)$, then put $s := s_{\text{new}}$ and go to step 5. Otherwise, the spherical cap with the center $O_i' = s_{\text{new}}$ and the radius $R_i = \min_{s_{\text{new}}, s_{\text{old}}} \rho(s_{\text{new}}, \partial V_i)$ is the optimal for the cell $V_i$, then go to step 7.
Steps 3 - 6 are performed independently for each cell $V_i$.

Step 7: To ensure that spherical caps do not intersect each other, choose the minimum packing radius $R^* = \min_{i=1,n} R_i$.

Step 8: If $\rho(O_i, O_j^*) < \delta, i = 1, n$, where $\delta$ is required accuracy, go to step 9, otherwise, $O_i := O_j^*$ and go to step 2.

Step 9: If the radius found in the current iteration is larger than the previous one, it is saved as a solution to the problem. A new generation of initial positions is performed (Step 1). The algorithm is terminated when the specified number of generations is reached.

**Numerical Experiment**

In this section, we present some preliminary numerical results. The experiment is carried out on a personal computer with Intel (R) Core(TM) i5-3337U (1.8 GHz, 4CPUs, 6 GB RAM) configuration and operating system Windows 10. The algorithm is implemented in C# programming language using Visual Studio 2012.

**Example 1:** This example presents the best solutions for packing problems with spherical segments at $n = 20$ on a unit hemisphere $\Theta = \{(x, y, z) | x^2 + y^2 + z^2 = 1, z \geq 0\}$ in the case when the metric is Euclidean $f(x, y, z) = 1$.

The problem was solved using the developed software package by means of multiple runs with the packing algorithm. Positions of the centers of the spherical segment at $n = 20$.

$$O_{20} = \{-0.328; -0.091; 0.940\}; \{0.608; -0.421; 0.673\}; \{0.361; 0.639; 0.679\}; \{0.527; 0.065; 0.847\}; \{-0.313; 0.912; 0.263\}; \{-0.363; -0.600; 0.713\}; \{0.863; 0.221; 0.455\}; \{-0.740; -0.614; 0.274\}; \{0.129; -0.364; 0.922\}; \{-0.714; 0.205; 0.669\}; \{-0.339; 0.602; 0.722\}; \{0.048; 0.287; 0.957\}; \{-0.764; 0.586; 0.268\}; \{0.594; -0.762; 0.259\}; \{0.171; -0.795; 0.582\}; \{-0.959; -0.096; 0.268\}; \{-0.260; -0.928; 0.268\}; \{0.928; -0.270; 0.257\}; \{0.688; 0.680; 0.255\}; \{0.224; 0.940; 0.256\}\}
And the best radius \( R_{20} = 0.256 \). The projections of the hemisphere and the packed spherical segment on the Oxy plane are shown in Fig. 3a. The diagram Voronoi of respectively packed spherical segments is shown in Fig. 3b.

![Figure 3A. Projection of the hemisphere and packed spherical segments onto the Oxy plane in Example 1](image)

**Example 2:** It is required to solve the problem of packing by spherical segments on a unit sphere \( \Theta = \{(x, y, z) | x^2 + y^2 + z^2 = 1\} \) in the case when the metric is Euclidean \( f(x, y, z) = 1 \).

In this example, Table 1 shows a comparison of the results of the packing algorithm with the known result. Here \( n \) - is the number of packed spherical segments, \( R_{\text{bst}} \) – the best packing radius found from the packing algorithm, \( R_{\text{kw}} \) – known best packing radius, \( \Delta R = \frac{R_{\text{bst}} - R_{\text{kw}}}{R_{\text{kw}}} \times 100\% \) - radius deviation, \( t \) – the execution time of the packing algorithm (calculated in seconds) and the number of random generations of initial positions \( = 80 \). For \( n = 3, 4, 6, 12 \) see (Toth, 1943 & 1953), for \( n = 5, 7, 8, 9 \) see (Schutte, 1951), for \( n = 10, 11 \) see (Danzer, 1986), for \( n = 13, 14 \) see (Musi & Tarasov, 2012 & 2015), and for \( n = 24 \) see (Robinson, 1986). The results under known and unknown conditions are shown in Figs. 4 и 5.
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<th>n</th>
<th>$R_{ntr}$</th>
<th>$R_{str}$</th>
<th>$\Delta R(%)$</th>
<th>t (seconds)</th>
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Table 1. Comparison of unit sphere packing results with the best known results under the Euclidean metric
Source: Authors findings
Figure 4. Sphere packings with known conditions $n = 4, 6, 8, 10, 12, 14$. Source: Authors findings

Figure 5. Sphere packing with unknown conditions $n = 16, 17, 18, 19, 20, 40$. Source: Authors findings
Example 3: We consider the problem of packing a unit sphere
\[ \Theta = \{(x, y, z) | x^2 + y^2 + z^2 = 1\} \] a dynamic environment with the metric, in which the wave propagation velocity is \[ f(x, y, z) = 1 + \frac{1.5z}{1 + z^2}. \]

Spherical segments have radii that increase when the center is near the south pole and decrease when they go to the north pole, and all segments have the same radius (Fig. 6). The reason is that the function \[ f(x, y, z) = 1 + \frac{1.5z}{1 + z^2} \] is covariate in the range \( z \in [-1;1] \), therefore, the closer the \( z \) value approaches -1 (i.e., the closer the center is to the South Pole), the lower the \( f \) value, respectively, the wave propagation speed increases.

<table>
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<th>( t ) (seconds)</th>
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<td>20</td>
<td>0.3770</td>
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Table 2. Results of unit sphere packings with metric in the example 3
Source: Authors findings
Example 4: We consider the case of a dynamic environment with a metric, in which the wave propagation velocity is equal to \( f(x, y, z) = \frac{0.9}{1.1 - z^2} \) for packings of the unit sphere

\[ \Theta = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \} \].

It can be seen from the characteristics of the function \( f(x, y, z) = \frac{0.9}{1.1 - z^2} \) that if the value \( z \) approaches zero, the value of the function is less, but the minimum value is \( \frac{0.9}{1.1} = 0.81 \), and vice versa, the farther away \( f \), the faster the value increases and the value of the function increases the maximum value = 9. Therefore, if the center of the spherical segment is closer to the equator, then the segment will have a circle with a larger radius compared to the two poles (Fig. 7).

<table>
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Figure 6. Packings of the sphere by spherical segments in the non-Euclidean metric in example 3
Source: Authors finding
On the problem of the densest packing of spherical segments into a sphere

Table 3. Results of unit sphere packings with metric in the example 4
Source: Authors findings

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<td>286.92</td>
</tr>
<tr>
<td>20</td>
<td>0.6810</td>
<td>294.16</td>
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</table>

Figure 7. Packings of the sphere by spherical segments in the non-Euclidean metric in example 4.
Source: Authors findings

Conclusions

In this article, we considered the optimal packing problem for a sphere with spherical segments in the following formulation: the number of packed objects is given, and they are assumed to be equal. The optimality criterion is to maximize their geodesic radius under the condition that the packing elements do not intersect, i.e., caps can have only tangent points on
the boundary. The packing problem has been known for more than 400 years. It was I. Kepler who first considered it. This problem is well studied in the classical formulation when we deal with the circle packing on a plane or the ball packing in 3D space. However, relatively few works are known for the case when the container is surface, for example, a sphere. The research carried out, which provides for the use of a non-Euclidean metric that characterizes the measure of the distance of points, in general, as far as we know, is a pioneer. Meanwhile, such tasks have a specific practical value and arise, in particular, in the field of information technologies, communications, logistics, etc.

Without setting the goal of proving the optimality of packings found, we focused our efforts on developing a new heuristic algorithm. As a mathematical basis, we applied the spherical Voronoi diagram and the optical-geometric analogy. As a result, we proposed a method that allows us to find locally optimal solutions. Since we managed to find calculations for comparison only for the Euclidean metric, two out of four examples were devoted to this particular case. The numerical experiment has shown that the new method gives results somewhat yield to the best known, but the calculations are performed fairly quickly (no more than a few minutes on a conventional PC); deviations do not exceed 0.7%. Moreover, our method is more comprehensive than traditional geometric approaches because it allows us to consider non-Euclidean metrics. In this case, unfortunately, there was nothing to compare the results with, and we presented only a brief analysis of the properties of the constructed packings.

Further research in this direction may be associated with the complication of the type of surfaces for which the packing is constructed, but here the problem of the correct formulation of the problem arises. The authors also expect to apply the results obtained to solving applied problems in the fields of information technology, communications, and logistics.

References


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On the problem of the densest packing of spherical segments into a sphere


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